On the Second Zagreb Index of Triangular Cycle Graphs

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Abstract

This study investigates the second Zagreb index $M_2(G)$ of triangular cycle graphs, derived from the geometric arrangement of triangular numbers. The second Zagreb index, defined as the sum of the products of vertex degrees for adjacent vertices, provides insights into edge-based connectivity and branching patterns. By analyzing the structure of triangular cycle graphs, we classify edges based on vertex degrees ($2-2$, $2-3$, or $3-3$) and derive explicit formulas for the second Zagreb index. Our results, including the formula $M_2(T(n)) = 2n^2 + 20n - 37$ for $n \geq 3$, enhance the understanding of these graphs' topological properties and their applications in network analysis and chemical graph theory.

Keywords: Second Zagreb index, triangular numbers, edge classification

Introduction

Figurate numbers represent a fascinating intersection of arithmetic and geometry, defined as integers that can be represented through regular geometric arrangements of points. Dating back to the Pythagoreans, these numbers provide intuitive visual representations of mathematical patterns and sequences [1-3]. Among the various classes of figurate numbers, triangular numbers constitute one of the most fundamental sequences, formed by arranging points in equilateral triangular patterns with each successive row containing one more point than the previous row [4, 5].

The study of figurate numbers extends beyond their geometric representation, finding connections to diverse mathematical concepts including binomial coefficients, combinatorial analysis, and various number sequences. These connections have motivated researchers to explore figurate numbers from different mathematical perspectives, including graph theory, where the geometric arrangements naturally translate into graph structures [6, 7].

Triangular numbers, denoted as T_n, are defined by the formula:

$$T_n = \frac{n(n+1)}{2} \tag{1}$$

The sequence begins $1, 3, 6, 10, 15, 21, 28, \ldots$ with each number representing the sum of the first n positive integers. When visualized as arrangements of points, triangular numbers form equilateral triangular patterns that can be analyzed as graph structures.

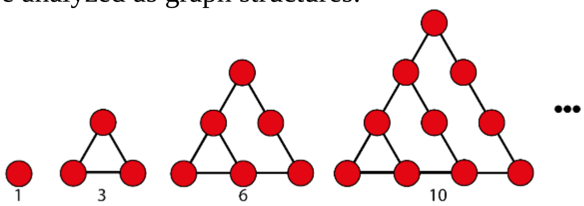

Figure 1. Triangular numbers

In graph theory, these triangular arrangements can be transformed into *triangular cycle graphs*, denoted as $T(n)$ for the n-th triangular number. These graphs are constructed by treating each point as a vertex and establishing edges between specific vertices according to defined adjacency rules [8, 9]. The vertices are typically labeled in a clockwise manner, facilitating theoretical analysis.

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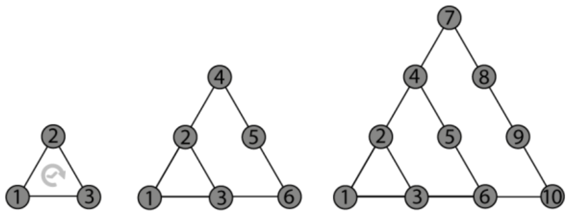

Figure 2. Some triangular cycle graphs $T(2)$, $T(3)$ and $T(4)$

Formally, a triangular cycle graph $T(n)$ has the vertex set: $V=\{1,2,3,\ldots,T_n\}$

The adjacency of vertices is defined as follows:

- $i \sim i+1$ unless i is a triangular number >1
- $T_i \sim T_{i+1}$ for $i=1,2,\ldots,n-1$
- $T_{i-1}+1 \sim T_i+1$ for $i=1,2,\ldots,n-1$

Topological indices are numerical parameters that characterize the structure of graphs, providing quantitative measures of various structural features. These indices have found significant applications in chemical graph theory, QSAR/QSPR studies, and network analysis [10, 11]. Among the most widely studied topological indices are the Zagreb indices, introduced by Gutman and Trinajstić [12].

While the first Zagreb index, $M_1(G)$, is calculated as the sum of the squares of vertex degrees, the second Zagreb index, $M_2(G)$, is defined as the sum of the products of degrees of adjacent vertices:

$$M_2(G)=\sum_{uv\in E(G)} d(u)\cdot d(v) \tag{2}$$

The second Zagreb index provides important insights into the edge connectivity patterns and distribution of degrees across adjacent vertices, making it a valuable tool for analyzing molecular structures and complex networks [13, 14].

Previous research by Demir et al. 2025 examined the first Zagreb index of triangular cycle graphs, deriving explicit formulas both in terms of triangular numbers and recursively [15]. Building upon this foundation, our study focuses on the second Zagreb index of triangular cycle graphs, seeking to establish similar mathematical relationships and expand the understanding of these graph structures.

Main Results

In this section, we analyze the second Zagreb index of triangular cycle graphs $T(n)$. The second Zagreb index, given by the sum of products of degrees of adjacent vertices, provides important information about the connectivity patterns and structural characteristics of these graphs.

To calculate the second Zagreb index, we need to identify the degree of each vertex and determine all adjacent vertex pairs. As established in previous research by Demir et al. 2025, vertices in triangular cycle graphs have degrees of either 2 or 3 [15]. Before proceeding to the main results, we will identify the different types of edges in triangular cycle graphs based on the degrees of their endpoint vertices.

Edge Classification: Let $n>1$ be a natural number. Then, in the triangular cycle graph $T(n)$, the edges can be classified into three types based on the degrees of their endpoint vertices:

- ❖ **Type A:** Edges connecting two vertices of degree 2, denoted as $(2,2)$ - edges.
- ❖ **Type B:** Edges connecting a vertex of degree 2 and degree 3, denoted as $(2,3)$ - edges**.**
- ❖ **Type C:** Edges connecting two vertices of degree 3, denoted as $(3,3)$ - edges.

Lemma 1. For $n\geq 1$, the number of edges in $T(n)$ is [15]:

$$E(T(n))=T_n+(n-2). \tag{3}$$

Lemma 2. Let $n\geq 3$ be a natural number. Then, in the triangular cycle graph $T(n)$:

- ❖ **Type A:** The number of $(2,2)$ - edges is: $N_A(n)=2+\frac{(n-2)(n-3)}{2}$.
- ❖ **Type B:** The number of $(2,3)$ - edges is: $N_B(n)=2(n-1)$.
- ❖ **Type C:** The number of $(3,3)$ - edges is: $N_C(n)=2n-5$.

Proof: We prove the lemma using mathematical induction on n.

1. Base Case; ($n=3$) For $n=3$, the formulas give:

- $N_A(3)=2+\frac{(3-2)(3-3)}{2}=2+\frac{1\cdot 0}{2}=2$.
- $N_B(3)=2(3-1)=2\cdot 2=4$.
- $N_C(3)=2(3)-5=6-5=1$.

The triangular cycle graph $T(3)$ has vertices $V=\{1,2,3,4,5,6\}$. The triangular numbers involved are $T_1=1,\ T_2=3,\ T_3=6$. The degrees of the vertices in $T(3)$ are: $d(1)=2,\ d(2)=3,\ d(3)=3,\ d(4)=2,\ d(5)=2,\ d(6)=2$.

The edges and their classifications based on the degrees of their endpoints are:

- $(1,2)$: $d(1)=2, d(2)=3 \Rightarrow$ Type B $(2,3)$ - edge.
- $(1,3)$: $d(1)=2, d(3)=3 \Rightarrow$ Type B $(2,3)$ - edge.
- $(2,3)$: $d(2)=3, d(3)=3 \Rightarrow$ Type C $(3,3)$ - edge.
- $(2,4)$: $d(2)=3, d(4)=2 \Rightarrow$ Type B $(2,3)$ - edge.
- $(3,6)$: $d(3)=3, d(6)=2 \Rightarrow$ Type B $(2,3)$ - edge.
- $(4,5)$: $d(4)=2, d(5)=2 \Rightarrow$ Type A $(2,2)$ - edge.
- $(5,6)$: $d(5)=2, d(6)=2 \Rightarrow$ Type A $(2,2)$ - edge.

Counting these edges by type:

- Number of Type A edges: 2 (edges $(4,5)$ and $(5,6)$). This matches $N_A(3)$.
- Number of Type B edges: 4 (edges $(1,2),(1,3),(2,4),(3,6)$). This matches $N_B(3)$.
- Number of Type C edges: 1 (edge $(2,3)$). This matches $N_C(3)$.

Thus, the formulas hold for the base case $n=3$.

2. Inductive Hypothesis; Assume that for some integer $k\geq 3$, the formulas hold for $T(k)$:

- ❖ $N_A(k)=2+\frac{(k-2)(k-3)}{2}$
- ❖ $N_B(k)=2(k-1)$
- ❖ $N_C(k)=2k-5$

3. Inductive Step; We need to show that the formulas hold for $T(k+1)$. The target formulas for $T(k+1)$ are:

- ❖ $N_A(k+1)=2+\frac{((k+1)-2)((k+1)-3)}{2}=2+\frac{(k-1)(k-2)}{2}$
- ❖ $N_B(k+1)=2((k+1)-1)=2k$
- ❖ $N_C(k+1)=2(k+1)-5=2k-3$

When transitioning from $T(k)$ to $T(k+1)$:

- **New Vertices:** The $(k+1)$-th row is added, consisting of $k+1$ vertices, denoted $v_{k+1,c}=T_k+c$ for $c=1,\ldots,k+1$. All these $k+1$ new vertices have a degree of 2 in $T(k+1)$. (Specifically, $v_{k+1,1}$ and $v_{k+1,k+1}=T_{k+1}$ are the degree - 2 end vertices of the last row, and $v_{k+1,c}$ for $1<c<k+1$ are degree-2 internal vertices of the last row, existing if $k+1\geq 3$.)
- **Vertex Degree Changes:** Certain vertices from $T(k)$ change their degrees in $T(k+1)$:
  - Vertex T_k (last vertex of row k): Its degree changes from $d_{T(k)}(T_k)=2$ (as the last vertex of $T(k)$) to $d_{T(k+1)}(T_k)=3$ (as row k is an intermediate row in $T(k+1)$ for $k\geq 3$).
  - Vertex $v_{k,1}=T_{k-1}+1$ (first vertex of row k): Its degree changes from $d_{T(k)}(v_{k,1})=2$ (as the first vertex of the last row of $T(k)$) to $d_{T(k+1)}(v_{k,1})=3$ (as row k is an intermediate row in $T(k+1)$ for $k\geq 3$).

The degrees of other relevant vertices in $T(k+1)$ are: $d_{T(k+1)}(v_{k,k-1}) = 2$ and $d_{T(k+1)}(v_{k,2}) = 2$ (these are internal to row k, which is not the last row and $k \geq 3$). Also, $d_{T(k+1)}(T_{k-1}) = 3$ and $d_{T(k+1)}(v_{k-1,1}) = 3$ (as row $k-1$ is an intermediate row for $k \geq 3 \Rightarrow k-1 \geq 2$).

Let $\Delta N_A, \Delta N_B, \Delta N_C$ be the net changes in the number of edges of each type.

- **Newly Added Edges in $T(k+1)$:**
  - Edges within row $k+1$: $(v_{k+1,c}, v_{k+1,c+1})$ for $c = 1,\dots,k$. There are k such edges. Since all vertices $v_{k+1,c}$ have degree 2 in $T(k+1)$, these k edges are all Type A $(2,2)$- edges.

  Contribution: $\Delta N_A = +k$.
  - Edge (T_k, T_{k+1}): In $T(k+1)$, $d_{T(k+1)}(T_k) = 3$ and $d_{T(k+1)}(T_{k+1}) = 2$. This is a new Type B $(2,3)$- edge.

  Contribution: $\Delta N_B = +1$.
  - Edge $(v_{k,1}, v_{k+1,1})$: In $T(k+1)$, $d_{T(k+1)}(v_{k,1}) = 3$ and $d_{T(k+1)}(v_{k+1,1}) = 2$. This is a new Type B $(2,3)$- edge.

  Contribution: $\Delta N_B = +1$.
- **Edges Present in $T(k)$ Whose Type Changes in $T(k+1)$:**
  - Edge $(T_k, v_{k,k-1})$ (where $v_{k,k-1} = T_k - 1$):
    - In $T(k)$: $d_{T(k)}(T_k) = 2$. For $k \geq 3$, $v_{k,k-1}$ is an internal vertex of row k, so $d_{T(k)}(v_{k,k-1}) = 2$. This edge was Type A $(2,2)$.
    - In $T(k+1)$: $d_{T(k+1)}(T_k) = 3$ and $d_{T(k+1)}(v_{k,k-1}) = 2$. This edge becomes Type B $(2,3)$.

  Contribution: $\Delta N_A = -1$, $\Delta N_B = +1$.
  - Edge (T_k, T_{k-1}):
    - In $T(k)$: $d_{T(k)}(T_k) = 2$. For $k \geq 3$, $d_{T(k)}(T_{k-1}) = 3$. This edge was Type B $(2,3)$.
    - In $T(k+1)$: $d_{T(k+1)}(T_k) = 3$ and $d_{T(k+1)}(T_{k-1}) = 3$. This edge becomes Type C $(3,3)$.

  Contribution: $\Delta N_B = -1$, $\Delta N_C = +1$.
  - Edge $(v_{k,1}, v_{k,2})$:
    - In $T(k)$: $d_{T(k)}(v_{k,1}) = 2$. For $k \geq 3$, $v_{k,2}$ is an internal vertex of row k, so $d_{T(k)}(v_{k,2}) = 2$. This edge was Type A $(2,2)$.
    - In $T(k+1)$: $d_{T(k+1)}(v_{k,1}) = 3$ and $d_{T(k+1)}(v_{k,2}) = 2$. This edge becomes Type B $(2,3)$.

  Contribution: $\Delta N_A = -1$, $\Delta N_B = +1$.
  - Edge $(v_{k,1}, v_{k-1,1})$:
    - In $T(k)$: $d_{T(k)}(v_{k,1}) = 2$. For $k \geq 3$, $d_{T(k)}(v_{k-1,1}) = 3$. This edge was Type B $(2,3)$.
    - In $T(k+1)$: $d_{T(k+1)}(v_{k,1}) = 3$ and $d_{T(k+1)}(v_{k-1,1}) = 3$. This edge becomes Type C $(3,3)$.

  Contribution: $\Delta N_B = -1$, $\Delta N_C = +1$.
- **Total Net Changes:**
  - $\Delta N_A = (\text{new}) + (\text{changes from Type A}) = k + (-1) + (-1) = k - 2$.
  - $\Delta N_B = (\text{new}) + (\text{changes to Type B}) + (\text{changes from Type B}) = (1+1) + (1+1) + (-1-1) = 2$.
  - $\Delta N_C = (\text{changes to Type C}) = 1 + 1 = 2$.
- **Calculating $N_X(k+1)$ using the Inductive Hypothesis and Net Changes:**
  - For Type A edges: $N_A(k+1) = N_A(k) + \Delta N_A = \left(2 + \frac{(k-2)(k-3)}{2}\right) + (k-2) = 2 + \frac{(k-2)(k-1)}{2}$.

  This matches the target formula for $N_A(k+1)$.
  - For Type B edges: $N_B(k+1) = N_B(k) + \Delta N_B = 2(k-1) + 2 = 2k - 2 + 2 = 2k$.

  This matches the target formula for $N_B(k+1)$.
  - For Type C edges: $N_C(k+1) = N_C(k) + \Delta N_C = (2k-5) + 2 = 2k - 3$.

  This matches the target formula for $N_C(k+1)$.

Since the formulas hold for $n = k+1$, by the principle of mathematical induction, the formulas hold for all natural numbers $n \geq 3$.

Corollary 1. For the case $n = 1$, the following properties hold for the triangular cycle graph $T(1)$ (which consists of a single vertex) and are considered trivial due to its structure: number of type (2,2) edges $N_A(1) = 0$, number of type (2,3) edges $N_B(1) = 0$, and number of type (3,3) edges $N_C(1) = 0$. Second Zagreb index: $M_2(T(1)) = 0$.

Proof: The graph $T(1)$ is, by definition, a single isolated vertex (K_1). As such, it contains no edges. Consequently, the counts for any type of edge ($N_A(1), N_B(1), N_C(1)$) are all zero. The second Zagreb index is a sum over the edges of the graph. Since there are no edges in $T(1)$, this sum is empty and evaluates to zero. These results are evident from the fundamental structure of $T(1)$.

Corollary 2. For the case $n = 2$, the following properties hold for the triangular cycle graph $T(2)$ (which is a 3-cycle) and are considered trivial due to its structure: number of type (2,2) edges $N_A(2) = 3$ number of type (2,3) edges $N_B(2) = 0$, and number of type (3,3) edges $N_C(2) = 0$. Second Zagreb index: $M_2(T(2)) = 12$.

Proof: The graph $T(2)$ is a 3-cycle graph (C_3). In a C_3 graph, each of the three vertices has a degree of exactly 2. Based on this structure:

- All three edges in the graph connect two vertices, each of degree 2. Therefore, all edges are of Type (2,2), leading to $N_A(2) = 3$.
- Since no vertices have a degree other than 2 (specifically, no degree 3 vertices exist), there can be no Type (2,3) or Type (3,3) edges. Thus, $N_B(2) = 0$ and $N_C(2) = 0$.
- The second Zagreb index, $M_2(T(2))$, is calculated from its three edges. Each edge uv contributes $d(u) \cdot d(v) = 2 \cdot 2 = 4$ to the sum.

Therefore, $M_2(T(2)) = (2 \cdot 2) + (2 \cdot 2) + (2 \cdot 2) = 3 \times 4 = 12$. These results are evident from the well-known and simple structure of the C_3 graph, which $T(2)$ represents.

Theorem 1. For the second Zagreb index of the triangular cycle graph $T(n)$, for $n \geq 3$:

$$M_2(T(n)) = 2n^2 + 20n - 37 \tag{4}$$

Proof: The second Zagreb index, $M_2(G)$, of a graph G is defined as: $M_2(G) = \sum_{uv \in E(G)} d(u) \cdot d(v)$ where $d(u)$ and $d(v)$ are the degrees of the vertices u and v incident to the edge uv. In the triangular cycle graph $T(n)$, for $n \geq 3$, the vertices have degrees of either 2 or 3. The edges of $T(n)$ are classified into three types based on the degrees of their endpoint vertices:

- **Type A:** Edges connecting two vertices of degree 2 ($(2,2)$- edges). The contribution of each such edge to $M_2(T(n))$ is $2 \cdot 2 = 4$.
- **Type B:** Edges connecting a vertex of degree 2 and a vertex of degree 3 ($(2,3)$ - edges). The contribution of each such edge to $M_2(T(n))$ is $2 \cdot 3 = 6$.
- **Type C:** Edges connecting two vertices of degree 3 ($(3,3)$ - edges). The contribution of each such edge to $M_2(T(n))$ is $3 \cdot 3 = 9$.

According to Lemma 2. (which provides the counts for these edge types for $n \geq 3$):

We can express $N_A(n) = 2 + \dfrac{n^2 - 3n - 2n + 6}{2}$, $N_B(n) = 2n - 2$, and $N_C(n) = 2n - 5$.

The second Zagreb index can thus be calculated as: $M_2(T(n)) = 4 \cdot N_A(n) + 6 \cdot N_B(n) + 9 \cdot N_C(n)$.

Substituting the expressions for $N_A(n)$, $N_B(n)$, and $N_C(n)$:

$$M_2(T(n)) = 4\left(\frac{n^2 - 5n + 10}{2}\right) + 6(2n - 2) + 9(2n - 5)$$

Now, we simplify the expression:

$$\begin{aligned} M_2(T(n)) &= 2(n^2-5n+10)+(12n-12)+(18n-45) \\ &= 2n^2-10n+20+12n-12+18n-45 \\ &= 2n^2+(-10n+12n+18n)+(20-12-45) \\ &= 2n^2+(2n+18n)+(8-45) \\ &= 2n^2+20n-37. \end{aligned}$$

This result is valid for $n \geq 3$, as Lemma 2 is applicable for $n \geq 3$. Thus, the theorem is proven.

Corollary 3. Let $T(n)$ be the n-th triangular number. For $n \geq 3$, the second Zagreb index of the triangular cycle graph $T(n)$ can be expressed in terms of T_n and n as: $M_2(T(n)) = 4T_n + 18n - 37.$

Proof: From Theorem 1, we have the formula for the second Zagreb index of $T(n)$ for $n \geq 3$: $M_2(T(n)) = 2n^2 + 20n - 37$. The n-th triangular number is defined as $T_n = \dfrac{n(n+1)}{2}$. From this definition, we can write $n(n+1) = 2T_n$, which expands to $n^2 + n = 2T_n$. Rearranging this equation to express n^2 in terms of T_n and n, we get: $n^2 = 2T_n - n$. Now, substitute this expression for n^2 into the formula for $M_2(T(n))$:

$$\begin{aligned} M_2(T(n)) &= 2(2T_n - n) + 20n - 37 \\ &= 4T_n - 2n + 20n - 37 \\ &= 4T_n + 18n - 37 \end{aligned}$$

This gives the expression for $M_2(T(n))$ in terms of T_n and n, valid for $n \geq 3$.

Corollary 4. The second Zagreb index of the triangular cycle graph $T(n)$ satisfies the following recurrence relation for $n \geq 4$: $M_2(T(n)) = M_2(T(n-1)) + 4n + 18$.

Proof: From Theorem 1, the second Zagreb index for $T(k)$ when $k \geq 3$ is given by the formula: $M_2(T(k)) = 2k^2 + 20k - 37$. We want to find the difference $M_2(T(n)) - M_2(T(n-1))$.This requires the formula to be valid for both $T(n)$ and $T(n-1)$. Thus, we must have $n \geq 3$ and $n-1 \geq 3$, which implies $n \geq 4$. For $n \geq 4$: $M_2(T(n)) = 2n^2 + 20n - 37$, and

$$\begin{aligned} M_2(T(n-1)) &= 2(n-1)^2 + 20(n-1) - 37 \\ &= 2n^2 - 4n + 2 + 20n - 20 - 37 \\ &= 2n^2 + 16 - 55 \end{aligned}$$

Now, we compute the difference:

$$\begin{aligned} M_2(T(n)) - M_2(T(n-1)) &= (2n^2 + 20n - 37) - (2n^2 + 16n - 55) \\ &= (2n^2 - 2n^2) + (20n - 16n) + (-37 + 55) \\ &= 4n + 18 \end{aligned}$$

Therefore, for $n \geq 4$: $M_2(T(n)) = M_2(T(n-1)) + 4n + 18$.

We give second Zagreb indices of some triangular cycle graphs below.

Table 1. Zagreb indices of some triangular cycle graphs

n	T_n	**Type A** $(2,2)$ $N_A(n)$	**Type B** $(2,3)$ $N_B(n)$	**Type C** $(3,3)$ $N_C(n)$	$M_1(T(n))$	$M_2(T(n))$
1	1	0	0	0	0	0
2	3	3	0	0	12	12
3	6	2	4	1	34	41
4	10	3	6	3	60	75
5	15	5	8	5	90	113
6	21	8	10	7	124	155
7	28	12	12	9	162	201
8	36	17	14	11	204	251
9	45	23	16	13	250	305
10	55	30	18	15	300	363

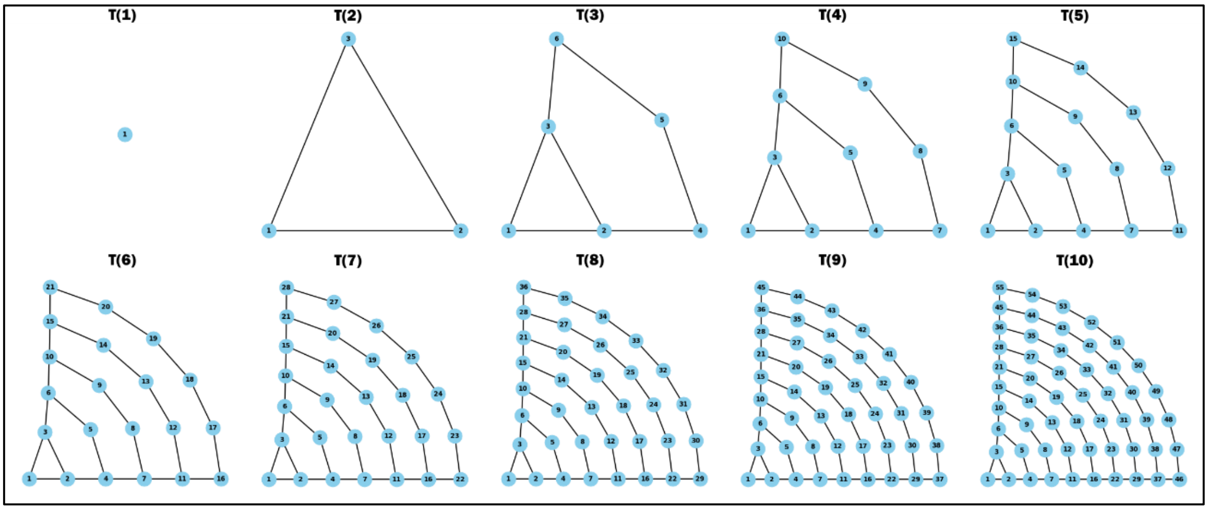

Figure 3. Some triangular cycle graphs by using PY-Codes

Conclusion

In this study, we have thoroughly investigated the second Zagreb index of triangular cycle graphs, a class of graphs derived from triangular numbers in figurate number theory. Building upon previous research on the first Zagreb index of these graphs, we have established explicit formulas for calculating the second Zagreb index, both in terms of the parameter n and through triangular numbers.

Our analysis began by classifying the edges in triangular cycle graphs based on the degrees of their endpoint vertices, identifying three distinct types: $(2,2)$ - edges, $(2,3)$ - edges, and $(3,3)$ - edges. Through mathematical induction, we derived formulas for calculating the number of each edge type in any triangular cycle graph $T(n)$, which subsequently led to the main result of our study: the second Zagreb index formula $M_2(T(n)) = 2n^2 + 20n - 37$.

Furthermore, we established alternative expressions for the second Zagreb index, including a formula in terms of triangular numbers $M_2(T(n)) = 4T_n + 18n - 37$. and a recursive relationship $M_2(T(n)) = M_2(T(n-1)) + 4n + 18$. These multiple representations provide flexibility in calculating the second Zagreb index and highlight the intricate mathematical patterns embedded within triangular cycle graphs.

The significance of this research extends beyond pure mathematical interest. The Zagreb indices, including the second Zagreb index studied here, have found valuable applications in chemical graph theory, particularly in predicting various properties of molecular structures. Our findings contribute to the broader understanding of these topological invariants and their relationship with graph structures derived from figurate numbers.

Future research directions could include exploring other topological indices for triangular cycle graphs, investigating the relationship between different Zagreb indices for these graphs, and extending the analysis to other classes of graphs derived from figurate numbers, such as square, pentagonal, or hexagonal cycle graphs.

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Appendix

PyCode: Polygonal Cycle Graph Construction & Visualization & Zagrep Indexes

```
import networkx as nx
import matplotlib.pyplot as plt
import math
def polygonal_cycle_graph(m, n):
    """ Generates the polygonal cycle graph for the nth m-gonal number. """
    def P_m(n):
        return n * ((m - 2) * n - m + 4) // 2
    V = list(range(1, P_m(n) + 1))
    G = nx.Graph()
    G.add_nodes_from(V)
    if n == 1:
        return G
    for i in V:
        if i > 1 and i < P_m(n) and not any(P_m(j) == i for j in range(2, n + 1)):
            G.add_edge(i, i + 1)
    for i in range(1, n):
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        G.add_edge(P_m(i), P_m(i + 1))
    for i in range(1, n):
        if P_m(i) + 1 < P_m(n):
            G.add_edge(P_m(i - 1) + 1, P_m(i) + 1)
    return G
def calculate_graph_metrics(m, n):
    G = polygonal_cycle_graph(m, n)
    num_edges = G.number_of_edges()
    M1, M2 = zagreb_indices(G)
    print(f"------------- PC_{m}({n}) -------------")
    print(f"Number of Edges: {num_edges}")
    print(f"Zagreb Index M1: {M1}")
    print(f"Zagreb Index M2: {M2}")
    visualize_graph(G, m, n) # Visualization
def zagreb_indices(G):
    M1 = 0
    M2 = 0
    for u, v in G.edges():
        deg_u = G.degree(u)
        deg_v = G.degree(v)
        M1 += deg_u + deg_v
        M2 += deg_u * deg_v
    return M1 , M2
def visualize_graph(G, m, n):
    plt.figure(figsize=(6, 6))
    pos = {}
    if n == 1:
        pos[1] = (0, 0)
    else:
        for i in range(1, n + 1):
            radius = i * 2
            if i == 1:
                angle_step = 0
            else:
                angle_step = 1/2 * math.pi / ((m - 2) * i)
            start_node = 1 if i == 1 else (i - 1) * ((m - 2) * (i - 1) - m + 4) // 2 + 1
            end_node = i * ((m - 2) * i - m + 4) // 2 + 1
            for j in range(start_node, end_node):
                pos[j] = (radius * math.cos(angle_step * (j - start_node)),
                          radius * math.sin(angle_step * (j - start_node)))
    nx.draw(G, pos, with_labels=True, node_size=700, node_color="skyblue",
            font_size=12, font_color="black", font_weight="bold",
            edge_color="black", width=2, linewidths=1)
    nx.draw_networkx_nodes(G, pos, node_size=700, node_color="skyblue")
    plt.title(f"PC_{m}({n}) Graph", fontsize=14)
    plt.axis('off')
    plt.show()
m = 3  # Triangle numbers
for n in range(1, 11):
    calculate_graph_metrics(m, n)
```