Numerical Techniques for Solving Linear Fractional Partial Differential Equations

*Bana Ali Raba**
University of N'Djamena, Department of Mathematics, N'Djamena, Chad
**Corresponding author: banaaliraba10@yahoo.com*

Abstract

Fractional partial differential equations appear in many fields, such as physics, finance, fluid mechanics, engineering, and biology. In this study, we focus on finding numerical solutions for linear fractional partial differential equations. We use the Homotopy Perturbation Method (HPM) to develop an approximate solution.

The goal of this study is to show that HPM is an efficient and reliable method for solving these equations. The fractional derivative is defined in the Caputo sense. The improved algorithm provides solutions as a series that quickly converges and is easy to compute. The results agree well with previous studies, confirming that this method is accurate, efficient, and simple to apply.

Keywords. Linear equations; approximation solution; Homotopy perturbation method; numerical method

Overview

Differential equations of fractional order have attracted considerable research attention in recent years because of their frequent appearance in a wide range of applications in fluid mechanics, viscoelasticity, biology, physics, and engineering. As a result, there has been a lot of interest in the solutions to integral and fractional differential equations of scientific significance Nayfeh, (1981); Miller, (1993). Since most fractional differential equations lack accurate analytic solutions, approximation and numerical approaches are necessary Momani and Odibat, (2006). He (1999) introduced the HPM, which is a new method for giving linear problems an analytical approximation. It is particularly helpful as a tool for scientists and applied mathematicians since it offers numerical approximation solutions to both linear and nonlinear differential equations without linearization or discretization, as well as instantaneous and visible symbolic expressions of analytical solutions. Because of this, HPM is a general entity that can solve a variety of linear and nonlinear problems.

This study aims to extend the estimate of the HPM to solve linear partial differential equations with a fractional derivative in time.

Fundamental Definitions

Definition 2.1. A real function $f(t), t > 0$ is said to be in the space C_α , $\alpha \in R$ if there exists a real number $p(p > \alpha)$, such that $f(t) = t^p f_1(t)$, where $f_1 \in C[0, \infty)$.

Definition 2.2. A real function $f(t), t > 0$ is said to be in the space C_α^m and $m \in N \cup \{0\}$ it is called whenever $f^m \in C_\alpha$.

Definition 2.3. The Riemann-Liouvill fractional integral operator of order $\alpha \geq 0$ of a function $f \in C_\alpha, \alpha \geq -1$, is defined as:

$$I^\alpha f(t) = \begin{cases} \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} d\tau \,, \quad \alpha > 0 \,, t > 0, \\ \\ f(t) \ , \alpha = 0 \end{cases} \tag{1}$$

Properties of the operator I^α can be found in Nayfeh, (1981); Miller, (1993). we mention only the following:

For $f \in C_\alpha, \alpha \geq -1, \beta, \gamma \geq 0$ and $\delta > -1$:

(1) $I^\beta I^\gamma f(t) = I^{\beta+\gamma} f(t)$,

(2) $I^{\beta}I^{\gamma}f(t) = I^{\gamma}I^{\beta}f(t)$,

(3) $I^{\beta}x^{\delta} = \frac{\Gamma(\delta+1)}{\Gamma(\beta+\delta+1)}x^{\beta+\delta}$

Definition 2.4. The fractional derivative of the function f in the Caputo form is presented as follows:

$$D_t^{\alpha}f(t) = I^{m-\alpha}D^m f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-\tau)^{m-\alpha-1} f^m(\tau)d\tau, \tag{2}$$

where

$$m-1 < \alpha \le m, \qquad m \in N, t > 0, f \in C_{-1}^m.$$

Cleary, a generalization of the above definition can be presented in the following:

$$D_t^{\alpha}f(t) = \begin{cases} \left[I^{m-\alpha}f^{(m)}(t)\right] & m-1 < \alpha < m, \quad m \in N, \\ \\ \frac{\partial^m}{\partial t^m}f(t) & \alpha = m. \end{cases} \tag{3}$$

Lemma 2.1. whenever $m-1 < \alpha \le m,\ m \in N,\ t > 0,\ f \in C_{-1}^m$, then

$$D_t^{\alpha}I^{\alpha}f(t) = f(t),$$

and

$$I^{\alpha}D_t^{\alpha}f(t) = f(t) - \sum_{k=0}^{m-1}\frac{\partial^k f(t)}{\partial t^k}\frac{t^k}{k!}.$$

Lemma 2.2. If $m \in N$ and the assumptions of the previous lemma hold, then

$$D_t^{\alpha}D_t^m f(t) = D_t^{\alpha+m}f(t),\ \ m = 0,1,\dots,\ \ n-1 < \alpha < n.$$

Examination of the Method

The fundamentals of the (HPM) and its applicability for various types of differential equations are described in Momani and Odibat (2006). For the simplicity of the reader, we will present a review of the (HPM) He, (2000); El-Shahed (2005), and then we will present the algorithm of the most recent modification of the (HPM). To complete our goal, we consider the nonlinear differential equation:

$$L(u) + N(u) = f(r), \quad r \in \Omega \tag{4}$$

considering the boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0\,, r \in \Gamma \tag{5}$$

where L is a linear operator, while N is a nonlinear operator, B is a boundary operator, Γ is the boundary of the domain Ω, and $f\ (r)$ is a known analytic function.

Using the HPM, He defines the homotopy $v(r,p)\colon \Omega \to \Omega \times [0,1] \to R$ which is content:

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[L(v) + N(v) - f(r)] = 0, \tag{6}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{7}$$

where $r\ \in\ \Omega, p \in [0,1]$ is an impeding parameter and u_0 is an initial approximation that satisfies the boundary conditions. It is clear from equations (6) and (7) that we have:

$$H(v,0) = L(v) - L(u_0) = 0, \tag{8}$$

$$H(v,1) = L(v) + N(v) - f(r) = 0. \tag{9}$$

The changing process of p from zero to unity is just of $v(r,p)$ from $u_0(r)$ to $u(r)$. In topology, this is called displacement $L(v) - L(u_0)$ and $L(v) + N(v) - f(r)$ are homotopic. The basic premise is that the solutions to equations (6) and (7) can be written as a power series in p.

$$v = v_0 + pv_1 + p^2v_2 + \cdots. \tag{10}$$

Equation (4) has an approximation of a solution when $p = 1$.

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{11}$$

In He (1999), and He et al. (2000). The convergence of the series (11) has been confirmed.

The Modified Homotopy Perturbation Technique

The homotopy perturbation technique, which gives an analytically approximate solution, is applied to different linear and nonlinear problems He et al. (2005). In this section, we present the algorithm for the novel modification of the HPM. To illustrate the basic ideas of the new modification, we consider the following nonlinear differential equation of fractional order:

$$D_t^\alpha u(x,t) = L(u,u_x,u_{xx}) + N(u,u_x,u_{xx}) + f(x,t),\ \ t > 0,\ \ m-1 < \alpha \le m, \tag{12}$$

where L is a linear operator, N is a nonlinear operator, which also might include other fractional derivatives of order less than α, f is a known analytic function, and $D_t^\alpha, m - 1 < \alpha \le m$, is the Caputo fractional derivative of order α, subject to the initial conditions.

$$\frac{\partial^k u(x,0)}{\partial t^k} = h_k(x), \quad k = 0,1,2,\dots,m-1 \tag{13}$$

In view of the homotopy technique, we can construct the following homotopy (Siddiqui, 2006).

$$\frac{\partial^m u}{\partial t^m} - L(u,u_x,u_{xx}) - f(x,t) = [\frac{\partial^m u}{\partial t^m} + N(u,u_x,u_{xx}) - D_t^\alpha u], \tag{14}$$

or

$$\frac{\partial^m u}{\partial t^m} - f(x,t) = p[\frac{\partial^m u}{\partial t^m} + L(u,u_x,u_{xx}) + N(u,u_x,u_{xx}) - D_t^\alpha], \tag{15}$$

where $p \in [0,1]$. The homotopy parameter p always changes from zero to unity.

When p = 0, equation (14) becomes the linearized equation:

$$\frac{\partial^m u}{\partial t^m} = L(u,u_x,u_{xx}) + f(x,t), \tag{16}$$

and equation (15) becomes the linearized equation:

$$\frac{\partial^m u}{\partial t^m} = f(x,t), \tag{17}$$

and when p = 1, equation (14) or equation (15) turns out to be the original equation (12). The fundamental premise is that the solution of equation (14) or equation (15) can be expressed as a power series in p.

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \cdots. \tag{18}$$

Thus, the series solutions of equation (12) are completely determined in (Momani and Odibat, 2007). The n-term approximate solution for equation (12) is:

$$u(x,t) = \textstyle\sum_{i=0}^{n-1} u_i(x,t) = u_0 + u_1 + \cdots + u_{n-1}. \tag{19}$$

In conclusion, we approximate the solution:

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t)$$

by the truncated series:

$$\Phi_N(x,t) = \textstyle\sum_{n=0}^{N-1} u_n(x,t). \tag{20}$$

Numerical Results

In order to evaluate the advantages and accuracy of the modified HPM for solving linear fractional differential equations, we have applied the method to an example of a differential equation of fractional order. All the results are calculated using the symbolic calculus software Mathematica.

Now we consider the linear fractional partial differential equation by Kamel Al-Khaled (2015).

$$D^{\alpha}_{*t}u = -x\frac{\partial u}{\partial x} - \frac{\partial^2 u}{\partial x^2} + 2(t^{\alpha} + x^2 + 1)\,,\ 0 < \alpha \le 1, \quad 0 \le x, t \le 1, \tag{21}$$

Subject to the initial condition

$$u(x,0) = x^2. \tag{22}$$

In view of equation (15), the homotopy for equation (21) can be constructed as:

$$\frac{\partial u}{\partial t} - 2(t^{\alpha} + x^2 + 1) \ = p\left[\frac{\partial u}{\partial t} - xu_x(x,t) - u_{xx}(x,t) - D^{\alpha}_{t}u(x,t)\right]. \tag{23}$$

Substituting (18) and the initial condition (22) into (23) and equating the terms with the same powers of p, we obtain the following set of linear partial differential equations:

$$p^0: \frac{\partial u_0}{\partial t} = 2(t^{\alpha} + x^2 + 1),\ \ u_0(0,x) = x^2,$$

$$p: \frac{\partial u_1}{\partial t} = \frac{\partial u_0}{\partial t} - x(u_0)_x - (u_0)_{xx} - D^{\alpha}_{t}u_0,\ \ u_1(x,o) = 0,$$

$$p^2: \frac{\partial u_2}{\partial t} = \frac{\partial u_1}{\partial t} - x(u_1)_x - (u_1)_{xx} - D^{\alpha}_{t}u_1,\ \ u_2(x,0) = 0,$$

Consequently, the first few components of the homotopy perturbation solution for equation (21) are derived as follows:

$$u_0(x,t) = x^2 + 2\left(t + tx^2 + \frac{t^{1+\alpha}}{1+\alpha}\right),$$

$$u_1(x,t) = -2t^2 - 2t^2x^2 + \frac{2t^{1+\alpha}}{1+\alpha} - \frac{t^{1-\alpha}x^2}{\Gamma(2-\alpha)} - \frac{2t^{2-\alpha}(1+x^2)}{\Gamma(3-\alpha)} - \frac{t^2\Gamma(2+\alpha)}{(1+\alpha)},$$

$$\begin{aligned}u_2(x,t) = {} & -2t^2 + \frac{4t^3}{3} - 2t^2x^2 + \frac{4t^3x^2}{3} + t^{-2\alpha}\left[\frac{t^2x^2}{\Gamma(3-2\alpha)} + \frac{2t^3(1+x^2)}{\Gamma(4-2\alpha)}\right] - \frac{t^{1-\alpha}x^2}{\Gamma(2-\alpha)} \\ & + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{2-\alpha}x^2}{(3-\alpha)} - \frac{2t^{2-\alpha}(1+x^2)}{\Gamma(3-\alpha)} + \frac{4t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{4t^{3-\alpha}x^2}{\Gamma(4-\alpha)} - \frac{2t^2\Gamma(2+\alpha)}{(1+\alpha)} \\ & + \frac{2t^{3-\alpha}(2 + 2x^2 + 2\alpha + 2x^2\alpha + \Gamma(2+\alpha))}{(1+\alpha)\Gamma(4-\alpha)},\end{aligned}$$

And so on. In the same manner, the rest of the components can be obtained using the Mathematica software. The three-order term approximate solution for equation (21) is given by:

$$\begin{aligned}u(x,t) = {} & -4t^2 + \frac{4t^3}{3} + x^2 - 4t^2x^2 + \frac{4t^3x^2}{3} + \frac{4t^{1+\alpha}}{1+\alpha} + 2\left(t + tx^2 + \frac{t^{1+\alpha}}{1+\alpha}\right) \\ & + t^{-2\alpha}\left[\frac{t^2x^2}{\Gamma(3-2\alpha)} + \frac{2t^3(1+x^2)}{\Gamma(4-2\alpha)}\right] - \frac{2t^{1-\alpha}x^2}{\Gamma(2-\alpha)} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + \frac{2t^{2-\alpha}x^2}{(3-\alpha)} \\ & - \frac{4t^{2-\alpha}(1+x^2)}{\Gamma(3-\alpha)} + \frac{4t^{3-\alpha}}{\Gamma(4-\alpha)} + \frac{4t^{3-\alpha}x^2}{\Gamma(4-\alpha)} - \frac{3t^2\Gamma(2+\alpha)}{(1+\alpha)} \\ & + \frac{2t^{3-\alpha}(2 + 2x^2 + 2\alpha + 2x^2\alpha + \Gamma(2+\alpha))}{(1+\alpha)\Gamma(4-\alpha)}\end{aligned}$$

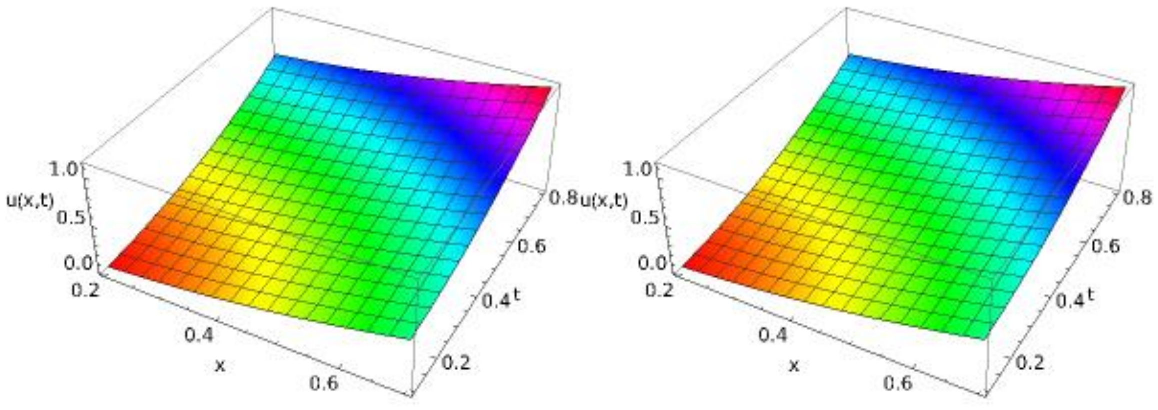
The Sumudu Decomposition Method (SDM) is also used in Kamel Al-Khaled et al. (2015), to solve the linear fractional partial differential equation. The decomposition series solution for equation (21)'s first three terms is provided as

$$u(x,t) = (x^2+1)\left[1+\frac{2t^{\alpha}}{\Gamma(1+\alpha)}\right] - (x^2+1)\left[\frac{2t^{\alpha}}{\Gamma(1+\alpha)}+\frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}\right] + (x^2+1)\left[\frac{4t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{8t^{3\alpha}}{\Gamma(1+3\alpha)}\right]$$

The table displays the approximations of the solutions to Eq. (21) obtained with HPM and SDM for various values of α. We only know the exact solution $u(x,t) = x^2 + \frac{2t^{2\alpha}\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}$, for values of $\alpha = 1$, and our approximate solution derived from the HPM is more accurate than the approximate solution derived from the SDM. It is to be noted that only the third-order term of the approximate solutions was used in Table 1.

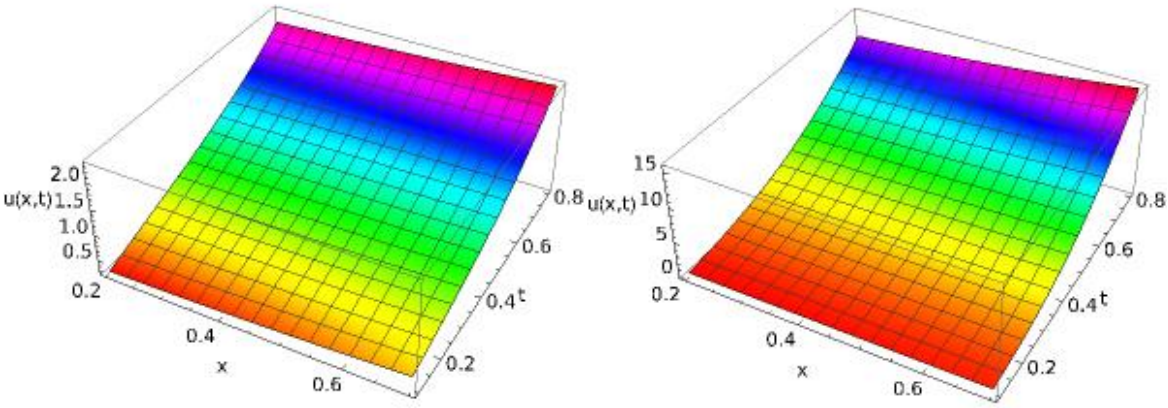
Table 1. Numerical values when $\alpha = 0.5,\ 0.75$ and 1.0 for Eq. (21)

t	x	$\alpha = 0.5$		$\alpha = 0.75$		$\alpha = 1.0$		α_{Exact}
		u_{HPM}	u_{SDM}	u_{HPM}	u_{SDM}	u_{HPM}	u_{SDM}	
0.1	0.2	0.094917	0.087965	0.064497	0.070008	0.052880	0.052991	0.050000
	0.4	0.589210	0.478962	0.278577	0.258751	0.175520	0.175961	0.170000
	0.5	0.613031	0.613531	0.465742	0.465892	0.259980	0.267896	0.260000
	0.7	0.686380	0.701548	0.587773	0.598745	0.500001	0.512078	0.500000
0.2	0.2	0.347758	0.447862	0.160529	0.200441	0.079093	0.078954	0.080000
	0.4	0.761297	0.775841	0.396577	0.501247	0.200000	0.200140	0.200000
	0.5	0.750530	0.699985	0.333325	0.333252	0.298933	0.287564	0.290000
	0.7	0.705456	0.705456	0.540771	0.582147	0.531893	0.530010	0.530000
0.5	0.2	1.042633	0.998741	0.550135	0.553510	0.291360	0.290000	0.290000
	0.4	0.770010	0.874125	0.671635	0.715423	0.410000	0.410000	0.410000
	0.5	0.882330	0.887542	0.607467	0.615487	0.499990	0.518721	0.500000
	0.7	1.999565	2.001475	0.987633	0.806587	0.741760	0.749875	0.740000
0.8	0.2	1.138250	1.325874	1.032855	0.998547	0.681920	0.678921	0.680000
	0.4	1.158382	0.985423	1.004910	1.004910	0.804368	0.810054	0.800000
	0.5	1.458607	1.498752	1.182504	1.206870	0.893200	0.895832	0.890000
	0.7	1.738922	1.738900	1.593134	1.600000	1.130719	1.134521	1.130000



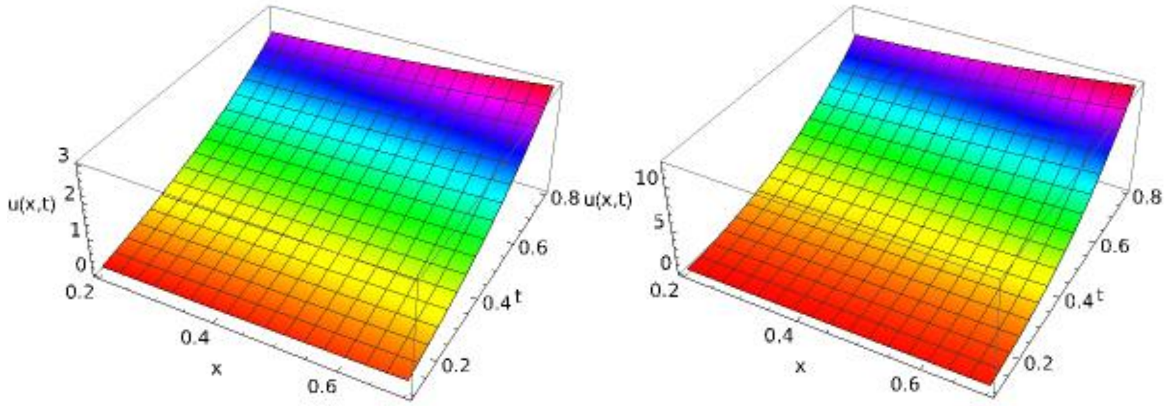
(a) (HPM) (b) (SDM)

Figure 1. Graphical simulation of the analytical solution when $\alpha = 1$.



(a) (HPM) (b) (SDM)

Figure 2. Graphical simulation of the approximation solution when $\alpha = 0.5$



(a) (HPM) (b) (SDM)

Figure 3. Graphical simulation of the approximation solution when $\alpha = 0.75$

Conclusion

The results show that the modified HPM described in this paper gives solutions as convergent series with simple algorithmic elements. The solution obtained using the HPM has very high accuracy compared with the SDM. Therefore, it is an effective and applicable technique for solving fractional-order linear partial differential equations.

References

El-Shahed, M. (2005). Application of He’s homotopy perturbation method to Volterra’s integro differential equation. Int J Nonlinear Sci Numer Simulat, 6(2), 163–8.

He J. H. (2005). Application of homotopy perturbation method to nonlinear wave equations. Chaos, Solitons & Fractals, 26(3), 695–700.

He, J. H. (2006). Homotopy perturbation method for solving boundary value problems, Phys. Lett. A 350(1-2) 87.

He, J. H. (2006). Some asymptotic methods for strongly nonlinear equations, Int. J. Mod. Phys. B 20(10) 1141.

He, J. H. (1999). Homotopy perturbation technique, Computer Methods in Applied Mechanics and Engineering (178) 257–262pp.

He. J. H. (1999). Variational Iteration method, A Kind of Nonlinear Analytical Technique: Some Examples, International Journal of Nonlinear Mechanics 34(4) 699–708p.

He. J.H. (2000). A coupling method of homotopy technique and perturbation technique for nonlinear problems, International Journal of Nonlinear Mechanics 35(1) 37–43pp.

Hemeda, A. A. (2012). Homotopy Perturbation Method for Solving Partial Differential Equations of Fractional Order, Int. Journal of Math. Analysis, Vol. 6, no. 49, 2431 – 2448p.

Kamel Al-Khaled. (2015). Numerical solution of time-fractional PDEs using Sumudu decomposition method, Rom. Journ. Phys., Vol 60, Nos. 1-2, P. 99–110, Boucharest.

Kamel Al-Khaled, & Momani, S. (2005). An approximate solution for a fractional diffusion-wave equation using the decomposition method. Applied Mathematics and Computation, 165(2), 473–483.

Miller, K.S., and Ross, B. (1993). An Introduction to the Fractional Calculus and Fractional, Differential Equations, John Wiley and Sons, Inc., ISBN (071)-58884-9pp.

Momani, S. (2006) Non- perturbative analytical solutions of the space- and time- fractional Burgers' equations, Chaos Solitons Fractals 28 (4) 930.

Momani, S., Odibat, Z. (2006) Analytical solution of a time- fractional NavierStokes equation by Adomian decomposition method, Appl. Math. Comput. 177 (2) 488.

Momani, S., Odibat, Z. (2007). Homotopy perturbation method for nonlinear partial differential equations of fractional order, Phys. Lett. A 365 345.

Momani, S., Odibat, Z. (2007). Numerical comparison of methods for solving linear differential equations of fractional order, Chaos Solitons Fractals 31 (5) 1248.

Nayfeh, A.H. (1981). Introduction to Perturbation Techniques, John Wiley & Sons. New York.

Odibat, Z., Momani, S. (2006). Application of variational iteration method to nonlinear equations of fractional order, Int. J. Nonlinear Sci. Numer. Simul. 1 (7) 15.

Odibat, Z., Momani, S. (2006). Approximate solutions for boundary value problems of time fractional wave equation, Appl. Math. Comput. 181 (1) (2006) 767.

Odibat, Z. Momani, S. (2008), Numerical methods for nonlinear partial differential equations of fractional order, Appl. Math. Mod. 32 28–39.

Siddiqui, A. Mahmood, R. Ghori, Q. (2006) Thin film flow of a third-grade fluid on a moving belt by He's Homotopy perturbation method, Int. J. Nonlinear Sci. Numer. Simul. 7(1) 7.